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Bounded sets in topological groups and embeddings ☆

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Abstract

We show that the existence of a non-metrizable compact subspace of a topological group G often implies that G contains an uncountable *supersequence* (a copy of the one-point compactification of an uncountable discrete space). The existence of uncountable supersequences in a topological group has a strong impact on bounded subsets of the group. For example, if a topological group G contains an uncountable supersequence and K is a closed bounded subset of G which does not contain uncountable supersequences, then any subset A of K is bounded in $G \setminus (K \setminus A)$. We also show that every precompact Abelian topological group H can be embedded as a closed subgroup into a precompact Abelian topological group G such that H is bounded in G and all bounded subsets of the quotient group G/H are finite. This complements Ursul's result on closed embeddings of precompact groups to pseudocompact groups.

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1. Introduction

In the article, we study the problems of when a topological group contains an uncountable supersequence and what the impact of the presence of such supersequences on bounded subsets of the group is. In Section 2 we show that if a topological group G contains a duplicate of uncountable compact space, or the two arrows space, or a copy of the ordinal space ω_1 with the order topology, then G contains an uncountable supersequence (that is, an uncountable compact set with a single non-isolated point). These facts follow from a more general result proved in Theorem 2.2, which is based on the concept of *fully closed mappings* defined and studied by Fedorchuk in [6,7].

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It is proved in Section 3 that if a paratopological group G contains a copy of the ordinal space $\omega_1 + 1$ or an uncountable supersequence then, for every sequence $S = \{y_n\colon n \in \omega\}$ converging to an element $g \in G$, the set S is bounded in $G \setminus \{g\}$. This result can be considerably strengthened in the case of topological groups: if a topological group G contains an uncountable supersequence then for every first countable (or metrizable) compact subset K of G and for every $S \subseteq K$, the set S is bounded in $G \setminus (K \setminus S)$ (see Corollary 3.4).

It is well known that every precompact Abelian topological group H can be embedded as a closed subgroup into a pseudocompact Abelian topological group G [14]. In particular, H is bounded in G. In Section 4 we go to the opposite direction and show that every precompact Abelian group H can be embedded as a closed subgroup into a precompact Abelian group G such that H is bounded in G and all bounded subsets of the quotient group G/H are finite.

1.1. Notation and terminology

All topological groups are assumed to be Hausdorff. A *supersequence* in a space is a compact subset with a single non-isolated point. Clearly, every supersequence is a one-point compactification of an infinite discrete set. A sequence $\{x_n\colon n \in \omega\}$ in a space X is said to be *trivial* if it is eventually constant.

A subset B of a space X is called *bounded* (or *functionally bounded*) in X if the image $f(B)$ is bounded in the real line $\mathbb{R}$, for every continuous function $f : X \to \mathbb{R}$. It is clear that pseudocompact subspaces of X are bounded, but not vice versa [8].

The character of a point x in a space X is denoted by $\chi(x, X)$ and, similarly, $\chi(F, X)$ is the character of a subset F of X, that is, the minimal cardinality of a base for X at F. The pseudocharacter of X is $\psi(X)$. The inequality $\psi(X) \leqslant \aleph_0$ holds if and only if each point of X is a G_δ-set in X.

A topological group G is *precompact* if G can be covered by finitely many translates of an arbitrary neighbourhood of the identity. Every pseudocompact group is precompact by a result in [2]. Similarly, a subset X of G is called *precompact* in G if for every neighbourhood U of the identity in G, one can find a finite subset F of X such that $X \subseteq FU \cap UF$. Every bounded subset of G is precompact in G, by [10, Assertion A].

If A is a subset of a group G then $\langle A\rangle$ is the minimal subgroup of G containing A. The kernel of a homomorphism $\pi : G \to H$ is denoted by $\ker\pi$. For an abstract group G and a non-empty set A, the direct sum of $|A|$ many copies of the group G is denoted by $G^{(A)}$. Then, for every non-empty set $B \subseteq A$, the group $G^{(B)}$ is naturally isomorphic to a subgroup of $G^{(A)}$.

A *paratopological group* is an abstract group with topology in which the multiplication is (jointly) continuous. The inversion in a paratopological group may fail to be continuous. Clearly, every topological group is paratopological, but not vice versa.

2. Supersequences in topological groups

The existence of certain type of compact sets in a topological group often implies that the group contains a long supersequence. Here we present a general result in this direction, Theorem 2.2, and deduce several corollaries. Our arguments make use of *fully closed mappings*.

Let $f : X \to Y$ be a continuous onto mapping. For every $U \subseteq X$, we put $f^\#(U) = Y \setminus f(X \setminus U)$. It is clear that if f is a closed mapping, then $f^\#(U)$ is open in Y for each open subset U of X. In addition, $f^\#(U)$ is the biggest subset of Y satisfying $f^{-1}f^\#(U) \subseteq U$, so that $f^\#(U) \subseteq f(U)$.

Following [6], we say that a continuous onto mapping $f : X \to Y$ is *fully closed* if for every point $y \in Y$ and every finite open cover $U_1, \dots, U_n$ of the fiber $f^{-1}(y)$ in X, the set $V_y = \{y\} \cup \bigcup_{i=1}^n f^\#(U_i)$ is a neighbourhood of y in Y. It is clear that every fully closed mapping is closed, so V_y must be open in Y. Fully closed mappings were defined and then applied by Fedorchuk to construct highly non-trivial examples of compact spaces with certain combinations of properties (see [5,7]).

The above definition of fully closed mappings is of local type. In what follows we will need a kind of *global property* of fully closed mappings which, in fact, characterizes fully closed mappings in the realm of compact spaces.

Let us say that a continuous mapping $f : X \to Y$ is *almost fully closed* if it satisfies the following condition:

(AFC) For every finite open cover $U_1, \dots, U_n$ of X, the complement $Y \setminus \bigcup_{i=1}^n f^\#(U_i)$ is closed and discrete in Y.

Lemma 2.1. *Let $f:X\to Y$ be a continuous mapping onto a regular space Y. Then*:

(a) *If f is fully closed, then it is almost fully closed.*
(b) *If f is closed and almost fully closed, then it is fully closed.*

Therefore, for closed mappings onto regular spaces, "almost fully closed" and "fully closed" coincide.

Proof. (a) Suppose that the mapping f is fully closed, and consider the complement $C=Y\setminus\bigcup_{i=1}^{n}f^{\#}(U_i)$, where $U_1,\dots,U_n$ is a finite open cover of X. Since f is closed, C is a closed subset of Y. Take an arbitrary point $y\in C$. Since the sets $U_1,\dots,U_n$ cover $f^{-1}(y)$, we have that $V_y=\{y\}\cup\bigcup_{i=1}^{n}f^{\#}(U_i)$ is an open neighbourhood of y in Y, and $V_y\cap C=\{y\}$. This implies that C is discrete.

(b) Suppose that f is a closed, almost fully closed mapping. Take an arbitrary point $y\in Y$ and consider a finite open cover $U_1,\dots,U_n$ of $f^{-1}(y)$ in X. Then $U=U_1\cup\dots\cup U_n$ is an open neighbourhood of the fiber $f^{-1}(y)$ in X and, since f is closed, there exists an open set V_0 in Y such that $y\in V_0$ and $f^{-1}(V_0)\subseteq U$. Take an open neighbourhood V_1 of y in Y such that $\overline{V_1}\subseteq V_0$, and put $U_{n+1}=f^{-1}(Y\setminus\overline{V_1})$. The sets $U_1,\dots,U_n,U_{n+1}$ are open and cover X, so $W=\bigcup_{i=1}^{n+1}f^{\#}(U_i)$ is open in Y and $C=Y\setminus W$ is closed and discrete. Therefore, the set $W_y=\{y\}\cup W$ is open in Y. It follows from $f^{\#}(U_{n+1})=Y\setminus\overline{V_1}$ that $y\in V_1\cap W_y\subseteq\{y\}\cup\bigcup_{i=1}^{n}f^{\#}(U_i)$. This in turn implies that $\{y\}\cup\bigcup_{i=1}^{n}f^{\#}(U_i)$ is an open neighbourhood of y in Y. We conclude that the mapping f is fully closed. □

Theorem 2.2. *Let $f:X\to Y$ be a fully closed mapping of countably compact spaces such that the set $T=\{y\in Y\colon |f^{-1}(y)|>1\}$ has cardinality $\tau\geqslant\omega$. Then every topological group containing X as a subspace also contains a supersequence of the length τ.*

Proof. Suppose that a topological group G with identity e contains X as a subspace. For every $y\in T$, choose distinct points $a_y,b_y\in f^{-1}(y)$. We claim that

$$S=\{e\}\cup\left\{a_y^{-1}b_y\colon\ y\in T\right\}$$

is a supersequence in G with the single non-isolated point e. It suffices to verify that for every neighbourhood O of e in G, the set $\{y\in T\colon\ a_y^{-1}b_y\notin O\}$ is finite.

Indeed, choose an open symmetric neighbourhood U of e such that $U^2\subseteq O$. Every countably compact subset of a topological group is bounded and, hence, precompact by [10, Assertion A]. So we can find points $x_1,\dots,x_n$ in X such that $X\subseteq\bigcup_{i=1}^{n}x_iU$. For every $i\leqslant n$, put $U_i=X\cap x_iU$. Then the sets $U_1,\dots,U_n$ form an open cover of X, so the complement $F=Y\setminus\bigcup_{i=1}^{n}f^{\#}(U_i)$ is finite, by Lemma 2.1. Take any point $y\in T\setminus F$. Then $y\in f^{\#}(U_i)$ for some $i\leqslant n$, so the points a_y and b_y lie in U_i. Hence, $a_y^{-1}b_y\in(x_iU)^{-1}x_iU=U^2\subseteq O$. This proves that $\{y\in T\colon\ a_y^{-1}b_y\notin O\}\subseteq F$, and our claim is proved.

It remains to verify that $|S|=|T|=\tau$. Suppose to the contrary that $|S|<|T|$. Then there exists an infinite set $Z\subseteq T$ and an element $g\in G$ such that $a_y^{-1}\cdot b_y=g$ for each $y\in Z$. It is clear that $g\neq e$. Take a neighbourhood O of e in G such that $g\notin O$. Then the set $\{y\in T\colon\ a_y^{-1}b_y\notin O\}$ is infinite, which contradicts the above claim.

We conclude, therefore, that S is a supersequence of the length τ in G. □

Our next step is to give several simple examples of fully closed mappings. Let X be an arbitrary space. Denote by X' a discrete copy of X disjoint from X, with a corresponding bijection $\varphi:X\to X'$, $\varphi(x)=x'$ for each $x\in X$. The *Alexandroff duplicate* of X is the union $A(X)=X\cup X'$, where all points of X' are isolated in $A(X)$ and a base of a point $x\in X$ in $A(X)$ consists of the sets $U\cup(\varphi(U)\setminus\{x'\})$, where U is an arbitrary open neighbourhood of x in X. If X satisfies one of the following properties, then so does $A(X)$: (a) the Hausdorff separation property; (b) regularity; (c) complete regularity; (d) normality; (e) compactness; (f) countable compactness; (g) the Lindelöf property (see [4, 3.1.26 and 3.1.G]).

Let $\psi:A(X)\to X$ be a mapping defined by $\psi(x)=x$ and $\psi(x')=x$, for each $x\in X$. It is clear that ψ is a continuous retraction of $A(X)$ onto its closed subspace X. An easy verification shows that the mapping ψ is perfect. The next simple result complements these facts.

Assertion 2.3. *The mapping $\psi:A(X)\to X$ is fully closed for every space X.*

Proof. Take an arbitrary point $x \in X$ and suppose that open sets U_1, U_2 in $A(X)$ cover the fiber $\psi^{-1}(x) = \{x, x'\}$, where $x \in U_1$. There exists an open neighbourhood V of x in X such that $V \cup (\varphi(V) \setminus \{x'\}) \subseteq U_1$. Then $V \setminus \{x\} \subseteq \psi^{\#}(U_1)$, whence it follows that $\{x\} \cup \psi^{\#}(U_1) \cup \psi^{\#}(U_2)$ is a neighbourhood of x in X. □

Let $X = C_0 \cup C_1$, where $C_0 = (0, 1] \times \{0\}$ and $C_1 = [0, 1) \times \{1\}$, be the *two arrows space* (see [4, 3.10.C]). It is well known that X is a perfectly normal compact Hausdorff space. Denote by f the mapping of X onto the closed unit interval $I = [0, 1]$ defined by $f(x, i) = x$, for all $x \in I$ and $i = 0, 1$. It is clear that f is continuous and, since X is compact, f is a closed mapping. This simple observation can be refined as follows:

Assertion 2.4. *The mapping $f : X \to I$ is fully closed.*

Proof. Let $U_1, \ldots, U_n$ be an open cover of X. We can assume that each U_i is a basic open set, that is, either $U_i = \{(x_i, 0)\} \cup (x_i - \varepsilon_i, x_i) \times \{0, 1\}$ or $U_i = \{(x_i, 1)\} \cup (x_i, x_i + \varepsilon_i) \times \{0, 1\}$, where $\varepsilon_i > 0$ for each $i \leqslant n$. In the first case, we have $f^{\#}(U_i) = (x_i - \varepsilon_i, x_i)$ and, in the second, $f^{\#}(U_i) = (x_i, x_i + \varepsilon_i)$. Hence $I \setminus \bigcup_{i=1}^{n} f^{\#}(U_i) \subseteq \{x_1, \ldots, x_n\}$. This together with Lemma 2.1 implies the assertion. □

Let $\gamma = \{\alpha\colon \alpha < \gamma\}$ be an infinite ordinal endowed with the order topology. Then the space γ is compact iff γ is a successor, i.e., $\gamma = \beta + 1$, and γ is countably compact iff either it is compact or if γ is limit and $\mathrm{cf}(\gamma) > \omega$. Let us define a mapping $g : \gamma \to \gamma$ by $g(\alpha) = \alpha$ if $\alpha \in \gamma$ is a limit ordinal and $g(\alpha + 1) = \alpha$ if $\alpha + 1 < \gamma$. In particular, $g(0) = 0$. It is easy to see that the mapping g is continuous and the image γ' of γ under g either coincides with γ (if γ is limit or if $\gamma = \beta + 1$ with β limit) or is equal to $\beta + 1$ if $\gamma = \beta + 2$. A direct verification shows that the mapping g is perfect. As in the previous cases, one can complement the latter fact:

Assertion 2.5. *The mapping $g : \gamma \to \gamma'$ is fully closed for each $\gamma \geqslant \omega$.*

Proof. Let $\alpha < \gamma'$ be arbitrary. If α is a successor, then $g^{-1}(\alpha) = \{\alpha\}$, and there is nothing to verify. The same happens if α is limit and $\gamma = \alpha + 1$. Suppose, therefore, that α is limit and $\alpha + 1 < \gamma$. Then $g^{-1}(\alpha) = \{\alpha, \alpha + 1\}$. Let U_1, U_2 be open sets in γ such that $\alpha \in U_1$ and $\alpha + 1 \in U_2$. Choose $\nu < \alpha$ such that the order interval $(\nu, \alpha]$ is contained in U_1. The definition of g implies that $(\nu, \alpha) \subseteq g^{\#}(U_1)$, whence it follows that $(\nu, \alpha] \subseteq \{\alpha\} \cup g^{\#}(U_1)$ is an open neighbourhood of the point α in γ'. Hence the mapping g is fully closed. □

The result proved in Assertion 2.5 can be extended to a wide subclass of linearly ordered spaces as follows. Let $(X, <)$ be a linearly ordered set. We endow X with the topology generated by the subbase consisting of the sets

$$(\leftarrow, x) = \{y \in X\colon\ y < x\} \quad \text{and} \quad (x, \rightarrow) = \{y \in X\colon\ x < y\},$$

where $x \in X$. Elements $x, y \in X$ with $x < y$ are said to be a *jump* in $(X, <)$ if no element $z \in X$ satisfies $x < z < y$. If $x, y \in X$ form a jump, the sets $(\leftarrow, x]$ and $[y, \rightarrow)$ are disjoint and clopen in the linearly ordered space $(X, <)$.

In what follows we assume that the linearly ordered space $(X, <)$ has no isolated points. In this case we say that the space $(X, <)$ is *crowded*. Consider the equivalence relation $\sim$ on X defined by $x \sim y$ if either $x = y$ or x, y is a jump in $(X, <)$. Our assumption implies that for each $x \in X$, the equivalence class $[x]$ in X contains at most two points. Let $\pi : X \to X_0$ be the natural projection of X onto the quotient set $X_0 = X/{\sim}$ defined by $\pi(x) = [x]$. We endow X_0 with the quotient topology. It is easy to verify that the mapping π is closed. Since the fibers of π have at most two points, π is a perfect mapping.

One can define a linear order $\prec$ on X_0 as follows. Suppose that $y_1 = [x_1]$ and $y_2 = [x_2]$ are distinct elements of X_0. Let $y_1 \prec y_2$ iff $x_1 < x_2$. This definition is correct since it does not depend on the choice of representatives x_1, x_2 in the classes y_1 and y_2. We leave to the reader a simple verification of the fact that the topology of the linearly ordered space $(X_0, \prec)$ and the quotient topology of X_0 coincide. Clearly, the linearly ordered space $(X_0, \prec)$ is also crowded.

Assertion 2.6. *The mapping $\pi : X \to X_0$ is fully closed.*

Proof. Take an arbitrary point $y \in X_0$. If $|\pi^{-1}(y)| = 1$, take $x \in X$ with $\pi(x) = y$ and use the fact that π is a closed mapping to deduce that $\pi^{\#}(U)$ is an open neighbourhood of y in Y, for each open neighbourhood of x in X. We can assume, therefore, that the fiber $\pi^{-1}(y)$ has two distinct elements x_1 and x_2 with $x_1 < x_2$. Then x_1, x_2 is a jump in $(X, <)$. Suppose that U_1 and U_2 are open sets in $(X, <)$ such that $x_i \in U_i$ for $i = 1, 2$. To avoid some trivialities, we can assume that neither x_1 nor x_2 is an end point of the linearly ordered set $(X, <)$. With this assumption, choose $t_1, t_2 \in X$ such that $(t_1, x_1] \subseteq U_1$ and $[x_2, t_2) \subseteq U_2$. Since the space $(X, <)$ is crowded, there exist $s_1, s_2 \in X$ such that $t_1 < s_1 < x_1$ and $x_2 < s_2 < t_2$. It follows from the definition of the equivalence relation $\sim$ on X that, in the ordered space $(X_0, \prec)$, we have the inclusions $(z_1, y) \subseteq \pi^{\#}(U_1)$ and $(y, z_2) \subseteq \pi^{\#}(U_2)$, where $z_i = \pi(s_i)$ for $i = 1, 2$. Hence the set

$$(z_1, z_2) \subseteq \{y\} \cup \pi^{\#}(U_1) \cup \pi^{\#}(U_2)$$

is an open neighbourhood of y in X_0. This proves the assertion. □

The more jumps a crowded linearly ordered space $(X, <)$ contains the more non-trivial fibers the quotient mapping $\pi : X \to X_0$ has. We present below several well-known examples of crowded linearly ordered spaces with uncountably many jumps.

Example 2.7. (a) Let $(X, <)$ be an uncountable linearly ordered space. Consider the product $P = X \times D$, where $D = \{0, 1\}$, and endow P with the lexicographic order defined by $(x_1, x_2) \ll (y_1, y_2)$ iff either $x_1 < y_1$ or $x_1 = y_1$ and $0 = x_2 < y_2 = 1$. Then the linearly ordered space $(P, \ll)$ has uncountably many jumps of the form $(x, 0)$ and $(x, 1)$, where $x \in X$. Notice that if $(X, <)$ is crowded, then so is $(P, \ll)$.

(b) Consider the set ${}^{\gamma}D$ of all functions from an ordinal $\gamma \geqslant \omega$ to $D = \{0, 1\}$. We define a linear order on ${}^{\gamma}D$ by letting $f \prec g$ for $f, g \in {}^{\gamma}D$ if there exists $\alpha < \gamma$ such that $f(\nu) = g(\nu)$ for all $\nu < \alpha$ and $0 = f(\alpha) < g(\alpha) = 1$. Clearly, the space $({}^{\gamma}D, \prec)$ is crowded. Notice that if γ is a successor, say, $\gamma = \beta + 1$ then the linearly ordered sets $({}^{\gamma}D, \prec)$ and $(({}^{\beta}D, \prec) \times D, \ll)$ are isomorphic, where the second space was defined in item (a). Therefore, we assume that γ is a limit ordinal.

Let us verify that the space $({}^{\gamma}D, \prec)$ contains many jumps. Indeed, suppose that $f, g \in {}^{\gamma}D$ is a jump. Then f and g are eventually constant. Therefore, there is $\alpha < \gamma$ such that $f(\alpha) = 0$ and $f(\beta) = 1$ for each β with $\alpha \leqslant \beta < \gamma$ and, similarly, g satisfies $g(\nu) = f(\nu)$ if $\nu < \alpha$, $g(\alpha) = 1$ and $g(\beta) = 0$ if $\alpha < \beta < \gamma$. A direct calculation shows that the number of jumps in $({}^{\gamma}D, \prec)$ is equal to $\sum_{\alpha<\gamma} 2^{|\alpha|}$. In particular, for any γ with $\omega < \gamma \leqslant \omega_1$ (limit or successor), the number of jumps in $({}^{\gamma}D, \prec)$ is equal to 2^{ω}.

Proposition 2.8. *Suppose that a topological group G contains one of the following sets*:

(a) *duplicate of an uncountable compact space*;
(b) *two arrows space*;
(c) *a copy of ω_1 with the order topology*;
(d) *a copy of the lexicographically ordered space $(X \times D, \ll)$, where $(X, <)$ is an uncountable crowded linearly ordered space* (*see* (a) *of Example* 2.7);
(e) *a copy of the linearly ordered space $({}^{\gamma}D, \prec)$, where $\gamma > \omega$* (*see* (b) *of Example* 2.7).

Then G contains an uncountable supersequence. In fact, in the cases (b) *and* (e), *G contains a supersequence of length* 2^{ω}.

Proof. Items (a), (b) and (c) follow from Assertions 2.3, 2.4, and 2.5, respectively, combined with Theorem 2.2. To deduce (d) and (e), it suffices to apply Assertion 2.6 along with Example 2.7 and conclude the argument with Theorem 2.2. □

Remark 2.9. A topological group G containing a copy of the ordinal space ω_1 may fail to contain a copy of $\omega_1 + 1$. Indeed, the free topological group $F(\omega_1)$ over the space ω_1 contains it as a closed subspace (the set of generators). By [11, Corollary 5], the group $F(\omega_1)$ is sequential and, therefore, cannot contain a copy of $\omega_1 + 1$.

3. Bounded subsets of topological groups

Let ω_1+1 be the space of ordinals $\leqslant \omega_1$ with the order topology and $\omega+1$ be the usual sequence converging to the point ω. We identify the non-isolated points ω_1 and ω of the spaces ω_1+1 and $\omega+1$, respectively, thus obtaining a space X in which the two "sequences" ω_1 and ω converge to the same point $x_0 \in X$. Clearly, ω_1 is countably compact, so it remains bounded in $X \setminus \{x_0\}$. However, the copy of $\omega \subseteq \omega+1$ is a clopen discrete subspace of X, so it is not bounded in X. It turns out that, in paratopological groups, the situation changes substantially.

Proposition 3.1. *Suppose that a paratopological group G contains a copy of the ordinal space ω_1+1 or an uncountable supersequence. If a sequence $S=\{y_n\colon n\in\omega\}\subseteq G$ converges to an element $g\in G$, then S is bounded in $G\setminus\{g\}$.*

Proof. Let $K=\{e\}\cup\{x_\alpha\colon \alpha<\omega_1\}$ be a subspace of G homeomorphic to a supersequence of the length ω_1, with the unique non-isolated point e, the identity of G. We assume that $e\neq x_\alpha\neq x_\beta$ if $\alpha\neq\beta$. Since translations in a paratopological group are homeomorphisms, we can assume that $g=e$, that is, the sequence S converges to e. Clearly, we can also assume that $e\notin S$. Then $T=S\cup\{e\}$ is a one-point compactification of S.

For every $n\in\omega$, there can exist only one index $\alpha_n<\omega_1$ such that $x_{\alpha_n}\cdot y_n=e$. Take $\beta<\omega_1$ such that $\alpha_n<\beta$ for each $n\in\omega$. Then $L=\{e\}\cup\{x_\alpha\colon \beta\leqslant\alpha<\omega_1\}$ is a subspace of G homeomorphic to K. Let $P=L\times T$ be the product space. It is easy to verify that the subset $\{e\}\times S$ of the space $X=P\setminus\{(e,e)\}$ is bounded in X. Consider the multiplication mapping $f:G\times G\to G$, $f(x,y)=x\cdot y$ for all $x,y\in G$. Since f is continuous, the image $S=f(\{e\}\times S)$ is bounded in $f(X)$. Our choice of the ordinal β implies that $e\notin f(X)$, so S is bounded in $G\setminus\{e\}$.

Similarly, suppose that $\{e\}\cup\{x_\alpha\colon \alpha<\omega_1\}$ is a subspace of G homeomorphic to ω_1+1, where e plays the role of the point ω_1 in ω_1+1. Choose an ordinal $\beta<\omega_1$ as above and put $L=\{e\}\cup\{x_\alpha\colon \beta\leqslant\alpha<\omega_1\}$. Note that the subspace $X=(L\times T)\setminus\{(e,e)\}$ of the compact space $L\times T$ is pseudocompact, so the set $S=f(\{e\}\times S)$ is bounded in $f(X)\subseteq G\setminus\{e\}$. □

In the case of topological groups, the above result can be given a much sharper form (see Theorem 3.3). First, we need an auxiliary result.

Proposition 3.2. *Let K be a closed bounded subset of a topological group H, and let $A\subseteq K$. If K does not contain non-empty G_δ-subsets of the group H, then A is bounded in $H\setminus(K\setminus A)$.*

Proof. Suppose to the contrary that K does not contain any non-empty G_δ-subset of the group H, but A is unbounded in $X=H\setminus(K\setminus A)$. It follows from our assumptions about K that X is dense in H. Take a continuous real-valued function f on X such that the image $f(A)$ is unbounded in $\mathbb{R}$. By induction, we can choose a sequence $\{x_n\colon n\in\omega\}\subseteq A$ such that $|f(x_{n+1})|\geqslant|f(x_n)|+3$, for each $n\in\omega$. Since f is continuous, for every $n\in\omega$ there exists an open neighbourhood U_n of the identity e in H such that $|f(x)-f(x_n)|<1$ for all $x\in X\cap x_n\cdot U_n$. Then the family $\{X\cap x_n\cdot U_n\colon n\in\omega\}$ is discrete in X and, since X is dense in H, all accumulation points of the family $\{x_n\cdot U_n\colon n\in\omega\}$ in H lie in K. Choose a sequence $\{V_n\colon n\in\omega\}$ of open symmetric neighbourhoods of e in H satisfying $V_{n+1}^2\subseteq V_n\subseteq U_n$, for each $n\in\omega$. Then the sets V_n's satisfy $\overline{V}_{n+1}\subseteq V_n$, so $N=\bigcap_{n\in\omega}V_n=\bigcap_{n\in\omega}\overline{V}_n$ is a closed subgroup of type G_δ in H. Let $\pi:H\to H/N$ be the quotient mapping of H onto the left coset space H/N. The space H/N is submetrizable by [13, Proposition 3]; in particular, H/N has countable pseudocharacter. Clearly, $\pi(K)$ is a bounded subset of H/N. Since every submetrizable space is Dieudonné-complete, by [4, 8.5.13 (g)], the closure of $\pi(K)$ in H/N, say, C is compact.

Note that $\pi(x_n)\in\pi(K)\subseteq C$, for each $n\in\omega$. In addition, $\pi(x_k)\neq\pi(x_l)$ if $k\neq l$. Indeed, otherwise $x_k^{-1}x_l\in N$ for some $k,l\in\omega$ with $k<l$. But then $x_l\in x_k\cdot N\subseteq x_n\cdot V_k$, whence it follows that $|f(x_l)-f(x_k)|<1$. This contradicts our choice of the points x_n's. Therefore, the infinite subset $\{\pi(x_n)\colon n\in\omega\}$ of the compact set C has an accumulation point $p\in C$. Since the mapping π is open, the fiber $\pi^{-1}(p)$ is in the closure of the set $\bigcup_{n\in\omega}\pi^{-1}\pi(x_n)=\bigcup_{n\in\omega}x_n\cdot N$. Since K is closed in H and does not contains non-empty G_δ-subsets of H, the intersection $\pi^{-1}(y)\cap K$ is nowhere dense in $\pi^{-1}(y)$, for each $y\in H/N$. Hence, $\pi^{-1}(p)$ is in the closure of the set $(\bigcup_{n\in\omega}x_n\cdot N)\setminus K\subseteq X$. Note that $x_n\cdot N\subseteq x_n\cdot V_n\subseteq x_nU_n$, for each $n\in\omega$, so all accumulation points of the family $\{x_n\cdot N\colon n\in\omega\}$ lie in K. This proves that $\pi^{-1}(p)\subseteq K$, contradicting our assumption about K and finishing the proof. □

Theorem 3.3. *Suppose that a topological group H contains an uncountable supersequence. If K is a closed bounded subset of H which does not contain uncountable supersequences and $A \subseteq K$ is an arbitrary set, then A is bounded in $H \setminus (K \setminus A)$.*

Proof. Let $F \subseteq H$ be an uncountable supersequence. One can assume that F contains the identity e of H and e is the unique non-isolated point of F. By Proposition 3.2, it suffices to verify that if P is a non-empty G_δ-set in H, then $P \setminus K \neq \emptyset$. Clearly, P can be written as $P = \bigcap_{n\in\omega} U_n$, where each U_n is open in H. Suppose to the contrary that $P \subseteq K$ and pick a point $x \in P$. The complement $F_n = F \setminus x^{-1}U_n$ is finite for each $n \in \omega$, so the set $F \setminus x^{-1}P = \bigcup_{n\in\omega} F_n$ is countable. Hence $F' = F \cap x^{-1}P$ is an uncountable supersequence and $xF' \subseteq P \subseteq K$ is also an uncountable supersequence, thus contradicting our assumption about K. This proves the theorem. □

Corollary 3.4. *Suppose that a topological group G contains an uncountable supersequence. If a closed bounded subset K of G satisfies $\psi(K) \leqslant \aleph_0$ and $A \subseteq K$, then A is bounded in $G \setminus (K \setminus A)$.*

Remark 3.5. Suppose that a subgroup H of a topological group G contains an uncountable supersequence. Then H can contain a sequence S converging to an element of $G \setminus H$ such that S is unbounded in H. Indeed, let G be the Σ-product of ω_1 copies of the discrete group $\mathbb{Z}(2) = \{0, 1\}$, i.e., G is the subgroup of 2^{ω_1} which consists of all elements x with the property that $|\{\alpha < \omega_1\colon x(\alpha) = 1\}| \leqslant \aleph_0$. Let also H be the corresponding σ-product in 2^{ω_1}, that is,

$$H = \left\{x \in 2^{\omega_1}\colon \left|\left\{\alpha < \omega_1\colon x(\nu) = 1\right\}\right| < \aleph_0\right\}.$$

Then H is a dense subgroup of the Fréchet–Urysohn group G, so every point of $G \setminus H$ is a limit of a convergent sequence lying in H. For every $\alpha < \omega_1$, denote by x_α the element of H defined by $x_\alpha(\nu) = 1$ iff $\nu = \alpha$. Then the subspace $\{\bar{0}\} \cup \{x_\alpha\colon \alpha < \omega_1\}$ of H is a supersequence of the length ω_1, where $\bar{0}$ is the neutral element of G. Note that the group H is σ-compact, hence normal. Therefore, every closed subset of H is C-embedded in H. Since every sequence S in H converging to a point of $G \setminus H$ is closed in H, S is unbounded in H.

4. Special embeddings into topological groups

Let us say that a space X is *B-closed* if all bounded subsets of X are closed. It turns out that a topological group G is B-closed iff all bounded subsets of G are finite (see [1, Lemma 4.3]). According to Example 4.5 of [1], there exist a precompact topological group G and an infinite, closed, bounded subgroup N of G such that all bounded subsets of the groups N and G/N are finite. This implies, in particular, that extensions of topological groups do not preserve the property of being B-closed.

Our aim here is to generalize the above result and show that every precompact Abelian topological group H can be embedded into an appropriate precompact Abelian topological group G as a closed bounded subgroup in such a way that all bounded subsets of the quotient group G/H are finite. We start with a lemma.

Lemma 4.1. *Let $\kappa \geqslant \mathfrak{c}$ be a cardinal with $\kappa^\omega = \kappa$ and consider the direct sum $R = \mathbb{Z}^{(\kappa)}$ of κ copies of the group of integers $\mathbb{Z}$. Then there exists a Hausdorff topological group topology τ on R such that the group (R, τ) is precompact, B-closed, and has weight $\leqslant \kappa$. In addition, the topology τ on R can be chosen to satisfy the condition*:

(a) *if $f : (R, \tau) \to K$ is a continuous homomorphism to a second countable topological group K, then $|\ker f| = \kappa$.*

Proof. It is clear that $|R| = \kappa$. For every non-empty countable set $A \subseteq \kappa$, consider the natural projection $p_A : \mathbb{Z}^{(\kappa)} \to \mathbb{Z}^{(A)}$ and let $\mathcal{H}_A$ be the family of all homomorphisms of $\mathbb{Z}^{(A)}$ to the circle group $\mathbb{T}$. Denote by τ the coarsest topology on R that makes continuous all homomorphisms of the family

$$\mathcal{H} = \left\{h \circ p_A\colon A \subseteq \kappa,\ |A| \leqslant \omega,\ h \in \mathcal{H}_A\right\}.$$

It is easy to see that (R, τ) is a Hausdorff precompact topological group. In what follows we denote (R, τ) simply by R. For every countable $A \subseteq \kappa$, the topology of $\mathbb{Z}^{(A)}$ inherited from R coincides with the Bohr topology of $\mathbb{Z}^{(A)}$. Therefore, all bounded subsets of $\mathbb{Z}^{(A)}$ are finite (see [3, Theorem 1.1.3]).

Suppose that K is an infinite subset of R. Then there exists a countable set $A \subseteq \kappa$ such that the intersection $K \cap \mathbb{Z}^{(A)}$ is infinite. Take a continuous real-valued function f on $\mathbb{Z}^{(A)}$ such that the image $f(K \cap \mathbb{Z}^{(A)})$ is unbounded in the reals. Then the continuous function $g = f \circ p_A$ is unbounded on K, so all bounded subsets of R are finite. We have proved that the group R is B-closed. Finally, the weight of R does not exceed κ since $|\mathcal{H}| \leqslant \kappa$.

Finally, let us verify (a). Suppose that $f : (R, \tau) \to K$ is a continuous homomorphism to a first countable topological group K. Then the kernel of f is of type G_δ in (R, τ) and, since the family $\mathcal{H}$ generates the topology τ, we can find a countable subfamily $\{f_n\colon n \in \omega\} \subseteq \mathcal{H}$ such that $\bigcap_{n=0}^{\infty} \ker f_n \subseteq \ker f$. For every $n \in \omega$, there exist a countable set $A_n \subseteq \kappa$ and a homomorphism $h_n \in \mathcal{H}_{A_n}$ such that $f_n = h_n \circ p_{A_n}$. Let $A = \bigcup_{n=0}^{\infty} A_n$. Then the set $A \subseteq \kappa$ is countable. Denote by $p^A_{A_n}$ the natural homomorphic retraction of $\mathbb{Z}^{(A)}$ onto $\mathbb{Z}^{(A_n)}$, $n \in \omega$. Then the composition $g_n = h_n \circ p^A_{A_n}$ is an element of $\mathcal{H}_A$ and $f_n = g_n \circ p_A$, for each $n \in \omega$. Therefore,

$$\mathbb{Z}^{(\kappa \setminus A)} = \ker p_A \subseteq \ker f_n$$

for each $n \in \omega$, whence it follows that $\mathbb{Z}^{(\kappa \setminus A)} \subseteq \ker f$ and $|\ker f| = \kappa$. This finishes the proof. □

Theorem 4.2. *Let H be a precompact Abelian topological group. There exists a precompact Abelian topological group G which contains H as a closed subgroup and such that H is bounded in G and the quotient group G/H is B-closed.*

Proof. Let $\kappa = (w(H) \cdot \mathfrak{c})^\omega$. Then $w(H) \leqslant \kappa$ and $\kappa^\omega = \kappa$. Denote by $\tilde{H}$ the completion of the group H. Then the compact group $\tilde{H}$ is topologically isomorphic to a subgroup of $\mathbb{T}^\kappa$. In what follows we identify $\tilde{H}$ (and, hence, H) with a corresponding subgroup of $\mathbb{T}^\kappa$.

We are going to define G as a subgroup of the product group $\mathbb{T}^\kappa \times \mathbb{T}^\kappa$. First, let $N = \{\bar{0}\} \times H \subseteq \mathbb{T}^\kappa \times \mathbb{T}^\kappa$, where $\bar{0}$ is the neutral element of $\mathbb{T}^\kappa$. Let also $R = (\mathbb{Z}^{(\kappa)}, \tau)$ be the Hausdorff precompact topological group defined in Lemma 4.1. Since $w(R) \leqslant \kappa$, we can identify R with a subgroup of $\mathbb{T}^\kappa$. Our strategy is to construct an algebraic homomorphism $\varphi : R \to \mathbb{T}^\kappa$ satisfying certain conditions and define the group G as the sum $G = P + N$, where

$$P = \{(x, \varphi(x))\colon x \in R\}$$

is the graph of φ. A direct verification shows that $N = G \cap (\{\bar{0}\} \times \mathbb{T}^\kappa)$, so N is a closed subgroup of G. We claim that the quotient group G/N is B-closed, independently of our choice of φ. Indeed, let $\pi_1 : \mathbb{T}^\kappa \times \mathbb{T}^\kappa \to \mathbb{T}^\kappa$ be the projection onto the first factor and p_1 the restriction of π_1 to G. Then N is the kernel of p_1, so the topology of the quotient group G/N is finer than that of the projection of G to the first factor, i.e., of the group $\pi_1(G) = \pi_1(P) = R \subseteq \mathbb{T}^\kappa$. Since all bounded subsets of R are finite, the same holds for bounded subsets of G/N.

For every $A \subseteq \kappa$, let $\pi_A : \mathbb{T}^\kappa \to \mathbb{T}^A$ be the projection and $\bar{0}_A$ the neutral element of the group $\mathbb{T}^A$. To guarantee the boundedness of N in G, the homomorphism φ will be constructed to satisfy the following condition:

(∗) For every non-empty countable set $A \subseteq \kappa$ and every $x \in \mathbb{T}^A$, there exists $y \in R$ such that $\pi_A(y) = \bar{0}_A$ and $\pi_A(\varphi(y)) = x$.

Consider the family

$$\gamma = \{(A, x)\colon \emptyset \neq A \subseteq \kappa,\ |A| \leqslant \aleph_0,\ x \in \mathbb{T}^A\}.$$

It follows from $\kappa = \kappa^\omega$ that $|\gamma| = \kappa$, so we can write $\gamma = \{(A_\alpha, x_\alpha)\colon \alpha < \kappa\}$. One can define by recursion two sets $Y = \{y_\alpha\colon \alpha < \kappa\} \subseteq R$ and $Z = \{z_\alpha\colon \alpha < \kappa\} \subseteq \mathbb{T}^\kappa$ satisfying the following conditions for each $\alpha < \kappa$:

(i) $y_\alpha \notin \langle Y_\alpha \rangle$, where $Y_\alpha = \{y_\nu\colon \nu < \alpha\}$;
(ii) $\pi_{A_\alpha}(y_\alpha) = \bar{0}_{A_\alpha}$;
(iii) $\pi_{A_\alpha}(z_\alpha) = x_\alpha$.

Such a construction is possible in view of (a) of Lemma 4.1 which implies, in particular, that $|\pi_A^{-1}(\bar{0}_A) \cap R| = \kappa$ for each countable $A \subseteq \kappa$. Let us define $\varphi(y_\alpha) = z_\alpha$ for every $\alpha < \kappa$. Notice that the set Y is algebraically independent in R by (i), so we can extend φ to a homomorphism of $\langle Y \rangle$ to $\mathbb{T}^\kappa$. Since the group $\mathbb{T}^\kappa$ is divisible, φ admits an extension to a homomorphism of R to $\mathbb{T}^\kappa$ which is denoted by the same letter. Condition (∗) now follows from (ii) and (iii).

With this choice of φ, we define P as the graph of φ and put $G = P + N$. It remains to verify that N is bounded in G. It follows from $(*)$ that

$$(\pi_A \times \pi_A)(G) \supseteq (\pi_A \times \pi_A)(P) \supseteq \{\bar{0}_A\} \times \mathbb{T}^A,$$

for each countable set $A \subseteq \kappa$. It is also clear that

$$(\pi_A \times \pi_A)(N) = \{\bar{0}_A\} \times \pi_A(H) \subseteq \{\bar{0}_A\} \times \mathbb{T}^A.$$

In other words, for every countable $A \subseteq \kappa$, the set $(\pi_A \times \pi_A)(N)$ is contained in the compact subset $\{\bar{0}_A\} \times \mathbb{T}^A$ of $(\pi_A \times \pi_A)(G)$. Let f be a continuous real-valued function on G. Since the group G is precompact, it follows from [12, Assertion 3.7] that f depends on at most countably many coordinates or, equivalently, one can find a non-empty countable set $A \subseteq \kappa$ and a continuous real-valued function g on $(\pi_A \times \pi_A)(G)$ such that $f = g \circ (\pi_A \times \pi_A) \restriction_G$. Since $(\pi_A \times \pi_A)(N)$ is contained in a compact subset of $(\pi_A \times \pi_A)(G)$, the image $f(N) = g((\pi_A \times \pi_A)(N))$ is a bounded subset of the reals. Hence $N \cong H$ is bounded in G. $\square$

5. Open problems

Here we formulate two problems related to the results in Sections 3 and 4.

A complete topological group G cannot contain a closed copy of the ordinal space ω_1 since the closure of ω_1 in G is compact and hence is homeomorphic to $\omega_1 + 1$. The concept of completeness for paratopological groups is not as clear as that for topological groups (even if we have to distinguish between Weil complete and Raĭkov complete topological groups). In fact, there are distinct definitions of completeness for paratopological groups [9]. This gives rise to many problems; one of them is given below.

Problem 5.1. Does there exist a regular "complete" paratopological group which contains a closed copy of the ordinal space ω_1?

We do not know whether one can drop "Abelian" in Theorem 4.2:

Problem 5.2. Does the non-Abelian version of Theorem 4.2 hold?

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